Life of Pi

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Blending learning: Proving pi is irrational (using high school level calculus)

Function

- All you need is Pre-U math!
- Let a and b be positive integers such that $q := \dfrac{a}{b}$ is an **irreducible** rational number.
- Consider a function $f(x)$ with n being a positive integer:

$$f(x) := \frac{x^n(a - bx)^n}{n!}$$

  Here, $n!$ is n-**factorial**.
- All the coefficients of $n!f(x) = x^n(a - bx)^n$ are integers.

Warm Up

- For the rational number $q = \frac{a}{b}$, the function

$$f(x) := \frac{x^n(a - bx)^n}{n!} \tag{1}$$

  satisfies $f\left(q - x\right) = f(x)$.

- Proof

Differentiation

- In Pre-U math, first and second derivatives are written as

$$f'(x) = \frac{df(x)}{dx}, \quad f''(x) = \frac{d^2 f(x)}{dx^2}$$

- In general, we define the k-th derivative as $f^{(k)}(x) := \dfrac{d^k f(x)}{dx^k}$.
- For the function (1), $f^{(k)}(x)$ contains terms of the form $x^\alpha (a - bx)^\beta$.
- Example: $f^{(n)}(x)$ includes $\dfrac{n!}{n!}(a - bx)^n$, $\dfrac{n!}{n!}x^n(-1)^n b^n$, and terms of the form $x^\alpha (a - bx)^\beta$.
- Therefore, $f^{(n)}(0) = a^n$ and $f^{(n)}(q) = (-1)^n a^n$.

Exercises

- Define a function $g(x)$:

$$g(x) := f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

- Show that
  (A) if $h(x) := x^{\alpha}(a - bx)^{\beta}$ with $\alpha > 0$ and $\beta > 0$, then $h(0) = h(q) = 0$.
  (B) $(-1)^n f^{(2n)}(x) = c \cdot n! b^n$, where $c > 1$.
  (C) $g(0) > 0$ and $g(q) > 0$
  (D) $\frac{d}{dx}\left(g'(x)\sin x - g(x)\cos x\right) = g''(x)\sin x + g(x)\sin x = f(x)\sin x.$

Integration and Basic Trigonometry

- Integration is anti-derivative.

$$\int_{\alpha}^{\beta} \frac{d}{dx} h(x) dx = h(x) \Big|_{\alpha}^{\beta}$$

  It is called the **fundamental theorem of calculus**.

- Integration from 0 to q

$$\int_0^q f(x) \sin x \, dx = \left(g'(x) \sin x - g(x) \cos x \right) \Big|_0^q$$

- Now, suppose π is a rational number and let $q := \frac{a}{b} = \pi$.

- From trigonometry, $\sin 0 = \sin \pi = 0$, $\cos 0 = 1$, and $\cos \pi = -1$. Therefore,

$$\int_0^{\pi} f(x) \sin x \, dx = g(\pi) + g(0). \tag{2}$$

Proof by Contradiction

- From Exercise (C) we know that $g(\pi) + g(0)$ is a positive integer, implying that $\int_0^\pi f(x) \sin x \, dx > 0$.
- Now, for $0 < x < \pi$, because of the functional form of $f(x)$, (1), we have
$$0 < f(x) \sin x < \frac{\pi^n a^n}{n!}.$$
- When n approaches ∞, the integral (2) becomes 0.
- This is a contradiction. Hence, the $\pi = a/b$ assumption (hypothesis) fails to hold, implying that π cannot be a rational number.

“Life” of Pi is Eternal

3. https://www.angio.net/pi/digits.html

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